Detecting weak phase locking in chaotic system with dual attractors and ill-defined phase structure

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A quantitative approach was constructed for detecting phase locking in a chaotic system with complex attractor structure via stroboscopic method. We study the route to weak phase locking by analyzing the stroboscopic points. The onset of weak phase locking detected by using this statistical approach and the critical coupling strength calculated by Lyapunov exponent are matched well. Detailed structure of phase locking intensity is described by the Arnold tongue diagram.

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Phase locking of chaotic oscillators by external driving has been studied systematically [1]. In this topic, the researches usually analyze the coherence dynamics based on Arnold tongue diagram [2,3]. Recently the studies in experiment and in theory show that a proper periodic perturbation can force a chaotic oscillator to become phase coherent [4]. Besides, Zhao studied the limits to chaotic phase synchronization between the chaotic systems with double-scroll structure of the attractor [5,6]. So far, detecting the phase locking for phase-incoherent chaotic system is still an open question [7–9].

Traditionally, phase synchronization is defined as locking of the phases $\varphi_{1,2}$, $n\varphi_1 - m\varphi_2$=const. One of the most common methods to estimate the phase is Hilbert transform [10]. But this traditional method is not suitable for many researches. In the following reviews, hence, the researchers provided some modulated method for the system with noncoherent phase. In laser experiment, Gaussian filter method was applied to study the phase synchronization between two individual laser systems [11]. Rosenblum *et al.* [12] recommended an approach by estimating the average frequency of the observed signals. Chen defined the phase based on the idea of the curvature of an arbitrary curve [13–15]. In the meantime, the recurrence plot method was applied to study the synchronization relation between chaotic systems because the recurrence plot method is not counted by the phase angle [7]. For the real time study on an experimental data, the stroboscopic method is applied [16]. In the case of periodic driving, especially, phase locking in chaotic system is defined in the stroboscopic manner [17]. Furthermore, some limits of the analysis appear when we study a chaotic attractor with ill-defined phase, such as the double-scroll attractors system [5,18].

Theoretically, every time the periodic driver passes through its local maximum, one can measure the chaotic attractor and get a stroboscopic point (SP) [4,19]. If all these measurements are confined to a region in the phase plane, this means that the periodic driver locks the chaotic attractor. When the driving amplitude is increased, the stroboscopic points are concentrated correspondingly; this process is called stroboscopic point concentration (SPC). A general technique based on stroboscopic technique is applied to detect strong SPC in a chaotic system with noncoherent phase, such as funneling attractor and spiking oscillator [9]. In Pereira's work an analysis method is claimed to quantify the SPC.

The quantification of SPC with ill-defined phase is an important work because the dynamics around the boundary of weak phase locking domain is unclear. However, all analysis methods mentioned above still cannot be well applied on a chaotic system with multiple attractor. There are two restrictions to previous analyses for detecting chaotic phase locking. First, the phase topology of driven system should have steady torus structure or simple unstable periodic orbit so that one can clearly cut a Poincaré section to define the zero phase. Second, the strange attractor of chaotic driven system should be single structure or can be transformed to be single attractor, such as the Lorenz system [20]. These restrictions stunned many researchers studying such a problem because all analyses failed. Our work is to study an analysis approach that can detect phase locking strength in a chaotic system with dual attractors and ill-defined phase structure.

This study discusses a phase locking of a chaotic system with dual attractors that cannot be transformed into a simple form, such as phase coherence or single attractor. Our analysis approach for detecting SPC is determined by the stats of SP distribution (SPD). The SPD of a chaotic system is varied with modulation of the driving force. When the driving force is strong enough, the SPD will become localized. On the contrary, when the driving force is too weak to lock the system, the SPD will become broad. Besides, another problem needs to be considered for the selection of driving force. A harmonic driver, such as sinusoid, will change the attractor topology to be simple and phase coherent [4,9]. In order to keep the attractor topology, the driving force is chosen by a memory of periodic signal, which is recorded from the same system. This prescribed nonlinear driving force is stable and period one so that the influence of unstable-unstable crisis can be reduced [19].

A chaotic Chua circuit system with one attractor has been studied in experiment and in simulation. Baptista *et al.* [4] studied the Chua system with single well-defined phase attractor, which can be locked by a harmonic sine perturbation if the driving amplitude is high enough and the driving frequency is detuned. In this paper, we study a chaotic Chua circuit system with dual attractors. The equation of Chua circuit system [21] is described by following formula.

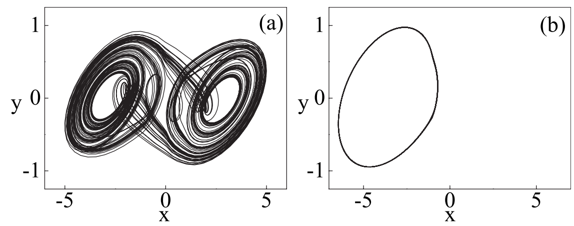

FIG. 1. The x-y phase diagram of (a) the chaotic attractor of Chua system at $R=1.65$ and (b) the periodic orbit of Chua system at $R=1.73$.

$$\dot{x} = 10(y-x)/R - 10g(x),$$

$$\dot{y} = (x-y)/R + z,$$

$$\dot{z} = -6y + f(t), \tag{1}$$

and

$$g(x) = -0.5x - 0.3(|x+1| - |x-1|).$$

The R is a parameter, the "$f(t)$" is an external driving signal, and x, y, and z are variables. Here, the "$g(x)$" is the effective nonlinear characteristic in this system [Eq. (1)]. The parameter R is reserved to two values corresponding to two different types of autonomous system [Figs. 1(a) and 1(b)]: the chaotic attractor and the period-one orbit. An output signal "z_0" generated by system (1) in period-one state has been memorized to be the driving force $f(t)$. By using a bifurcation diagram as the function of R, one can tell that the Chua system is chaotic at $R=1.65$ and the Chua system is periodic at $R=1.73$. The chaotic system in Fig. 1(a) has two attractors with ill-defined phase structure. Generating the periodic driving force in the simulation is like generating the signal by a function generator in the experiment. The driving force can be modulated by an amplitude modulation parameter or coupling strength, that is, "$f(t)=m_A z_0$." When the amplitude modulation is larger than the critical value, the chaotic system will be phase locked by the driving force. Nevertheless, this critical value does not only depend on the definition of phase locking but also the similarity between driving force and the system.

The SP in this study is defined by sampling the chaotic attractor every time the periodic driver passes through its maximum value of $z_0(t)$. In Fig. 2, the m_A is varied to show the route from uncoupling to weak SPC. When the amplitude modulation is zero, the phase locking in the chaotic attractor is impossible so that the SPs are spread randomly on this attractor domain [Fig. 2(a)]. As the amplitude modulation is increased over the critical strength, the SP will start to concentrate gradually [Figs. 2(b)–2(d)]. The amplitude modulation parameters in Fig. 2 are (a) $m_A=0$, (b) $m_A=0.005$, (c) $m_A=0.01$, and (d) $m_A=0.02$. In Fig. 2, the gray line is the phase plot of the chaotic attractor by x-y and the black point is a SP corresponding to different amplitude modulation. Parts of SP in Fig. 2(b) are assembled slightly to smaller area than the SP in Fig. 2(a). As the amplitude modulation $m_A=0.01$ [Fig. 2(c)], most parts of SP are confined to several groups. All SPs are highly concentrated as the amplitude modulation is increased up to $m_A=0.02$. Here, a small intensity of amplitude modulation is effective to confine the SP. The phase plot, however, shows that the phase structure in this chaotic system is still ill defined and complex when the weak phase locking appears. In other words, this periodic driving force can only lock the phase of the chaotic attractor without modifying the attractor's characteristic too much.

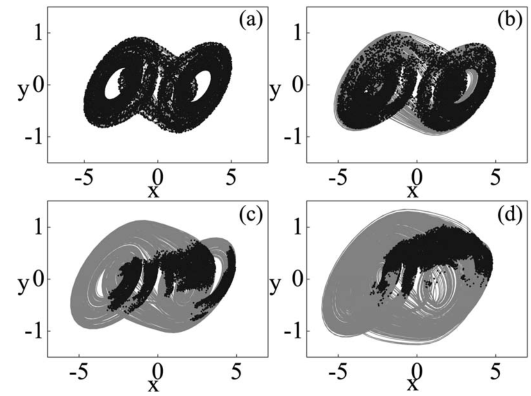

FIG. 2. The x-y phase diagram of chaotic attractor, which is driven by different amplitude modulation, is drawn in gray line, and their stroboscopic points in black. The amplitude modulation parameters are (a) $m_A=0$, (b) $m_A=0.005$, (c) $m_A=0.01$, and (d) $m_A=0.02$.

The dynamics of weak SPC depend on the amplitude modulation. This study uses a statistic approach to quantify the SPC behavior. The location of SP on phase diagram is defined by $\mathbf{x}_i=(x_i,y_i,z_i)$ for system (1) (the black points in Fig. 2). The relative distance between each pair of stroboscopic points is $\mathbf{r}_{ij}=\|\mathbf{x}_i-\mathbf{x}_j\|$, $i\neq j$, where the $\|\cdot\|$ denotes Euclidean norm. The stroboscopic point distribution is counted by the normalized probability of SP as the function of relative distance $\mathbf{r}$. The probability of SP $P_f(\mathbf{r})$ marks the chaotic attractor every time the periodic driver passes through its maximum value, where the suffix f means "forced." The surrogate analysis $P_r(\mathbf{r})$ is counted by the stroboscopic points that are sampled in random time, where the suffix r means "random." The $P_r(\mathbf{r})$ must distribute over all the orbit domain of the chaotic attractor. When amplitude modulation is zero, the distribution of $P_f(\mathbf{r})$ and $P_r(\mathbf{r})$ are the same as shown in Fig. 3(a). As amplitude modulation increased, the $P_f(\mathbf{r})$ becomes more localized than the $P_r(\mathbf{r})$. The distribution in Figs. 3(c) and 3(d) manifests the work showing that the driving force confines the stroboscopic points to several groups. In Fig. 3, the distance counts by 32 bins and the parameters are (a) $m_A=0$, (b) $m_A=0.005$, (c) $m_A=0.01$, and (d) $m_A=0.02$.

We quantify these distribution by the definition of the Shannon entropy, $H=-\Sigma_N P(\mathbf{r}_i)\log P(\mathbf{r}_i)$, where N is total number of bins and i is the state of relative distance. Thus the entropy of randomly marked SPD can be counted by $H_r=-\Sigma_N P_r(\mathbf{r}_i)\log P_r(\mathbf{r}_i)$, and the entropy of periodically marked SPD can be counted by $H_f=-\Sigma_N P_f(r_i)\log P_f(r_i)$. Without driving force, there is no phase locking between periodic driving force and the chaotic system [in Figs. 2(a) and 3(a)]; thus the result is $H_f\approx H_r$. With strong enough amplitude modulation, driving force can confine the forced SP to be

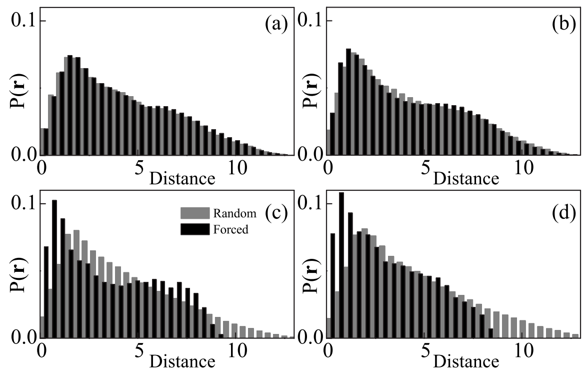

FIG. 3. The probability of stroboscopic point $P_f(\mathbf{r})$, which is sampled by periodic time, and $P_r(\mathbf{r})$, which is sampled by random time, as the function of relative distance is counted in the amplitude modulation: (a) m_A=0, (b) m_A=0.005, (c) m_A=0.01, and (d) m_A=0.02.

more localized than the random sampling SP [in Figs. 2(c) and 3(c)]; thus the result is $H_f > H_r$. By this relation, we quantify the phase locking strength by the definition: $E_{sp} = H_f / H_r$, where the E_{sp} is the entropy ratio of SPD.

The two largest Lyapunov exponent (LE) and the E_{sp} are calculated as the function of amplitude modulation. The positive largest LE [gray line in Fig. 4(a)] denotes that the force $f(t)$ is too weak to synchronize system (1). When the amplitude modulation is less than the critical value, which is marked by the black arrow in Fig. 4, the second large LE is still zero [black line in Fig. 4(a)] and the value of E_{sp} [Fig. 4(b)] is one. This critical value is the onset of weak phase locking. The onset value in Figs. 4(a) and 4(b) is matched well by statistic power. The route to weak phase locking can be described well by the intensity of E_{sp}. While the amplitude modulation is larger than the critical value, the second large LE starts to become negative and the E_{sp} starts to grow above one. After all, the entropy ratio of stroboscopic point distribution E_{sp} gives a good statistical approach to quantify the weak phase locking in chaotic system with ill-defined phase structure. This analysis approach is powerful enough

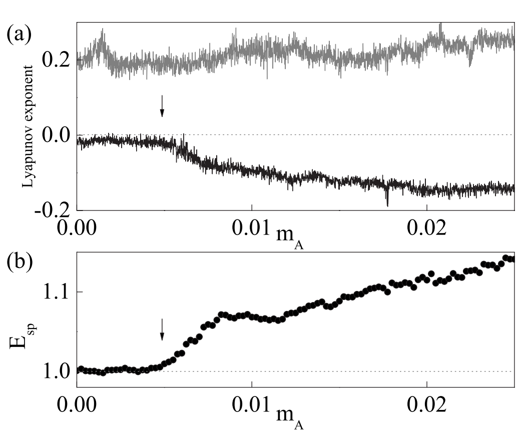

FIG. 4. (a) The largest two LE of the chaotic Chua system as the function of amplitude modulation m_A are drawn. The gray line shows the largest LE and the black line shows the second LE. (b) E_{sp} as the function of amplitude modulation m_A. The black arrow marks the onset of weak phase locking.

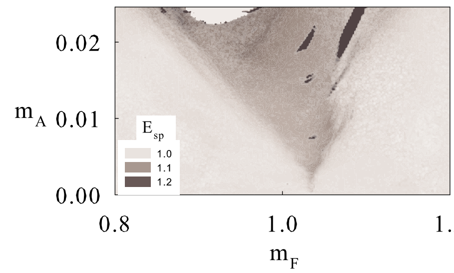

FIG. 5. (Color online) The E_{sp} intensity diagram is described as the functions of amplitude modulation m_A and frequency modulation m_F. The hemicircle area in white is the no solution domain.

to detect the onset of weak phase locking, in case the Lyapunov exponent of some dynamics systems cannot be calculated [6,20].

The dynamics of phase locking depend on the temporal condition of driving force. Some experimental results in review article show that a chaotic system has resonant response to frequency modulation. In order to detune the temporal parameter in periodic force, we define the driving force as $f(t) = m_A z_0(m_F t)$, where the m_F is a frequency modulation parameter. In Fig. 5, the E_{sp} intensity is calculated as the function of amplitude and frequency modulation. This Arnold tongue diagram is unlike the traditional one that states no information about intensity. For the constant amplitude modulation, the E_{sp} will respond as a resonant function to m_F. Besides, the local maximum E_{sp} and the boundary of weak phase locking region can be detected. The weak phase locking boundary and structure are both not symmetrical to frequency modulation. As the frequency modulation $m_A \approx 1.03$, the minimum onset of weak phase locking is detected in the tail of the tongue.

In conclusion, a chaotic system with complex phase attractors causes a difficult problem in detecting phase locking. We demonstrate a statistic approach to detect weak phase locking by counting the stroboscopic point marked by periodic timing in the chaotic Chua system with a nonlinear periodic driving force, which was generated by the Chua system itself. By using the definition of Shannon entropy, we calculate the distribution of SP in different modulations of the driving force. In the statistical result, the largest Lyapunov exponent is always positive; thus no synchronization can be observed. The originally null second largest Lyapunov exponent becomes negative because the amplitude modulation is larger than the onset value of weak phase locking. This analysis result by Lyapunov exponent matches well the analysis result based on the entropy ratio of stroboscopic point distribution. To apply this statistic approach on analyzing the weak phase locking intensity as the function of amplitude and frequency modulation, the boundary of weak phase locking is described. This statistic approach is powerful enough to apply on the other chaotic system with many attractors.

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